RESTRICTED RANDOM WALK WITH ONE BARRIER

By

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INTRODUCTION

Let X_i be a random variable associated with the i^{th} step of the particle s.t.

$$X_i = \begin{cases} +1, & \text{if the particle moves one unit step upward} \\ -1, & \text{if the particle moves one unit step downward} \end{cases}$$

with respective probabilities p and q(= 1-p).

Writing

$$S_i = X_1 + X_2 + \ldots + X_i \; ; i = 1, 2, \ldots, n; \; S_o = 0$$

then

$$S_i - S_{i-1} = X_i = \pm 1$$

When the points (i, S_i) are plotted on xy-plane and joined successively by straight line segments we get a path whose vertices have abscissa 0,1, . . ., n and ordinate $S_o, S_1, \ldots, S_n$ respectively. Such a path may be taken as representing the simple random walk.

Authors (1976) have investigaged certain results for this random walk starting at the origin and arriving at it at the 2nth step when it is restricted by a condition 'E' described below:

"If $S_r = 0$ for $r = 2\alpha_1, 2\alpha_2, \ldots, 2\alpha_i = 2n$, $i = 1,2,\ldots,n$ then the j^{th} segment included between the $(j-1)^{th}$ and j^{th} zeroes satisfy the following condition:

$$
\text{`E'}\begin{cases}
0 = S_{2\alpha_{j-1}} < S_{2\alpha_{j-1}+1} < \cdots < S_{\alpha_j-\alpha_{j-1}} > S_{\alpha_j-\alpha_{j-1}+1} \\
\qquad > S_{\alpha_j-\alpha_{j-1}+2} > \cdots > S_{2\alpha_j} = 0 \\
\qquad \text{or} \\
0 = S_{2\alpha_{j-1}} > S_{2\alpha_{j-1}+1} > \ldots > S_{\alpha_j-\alpha_{j-1}} < S_{\alpha_j-\alpha_{j-1}+1} \\
\qquad < S_{\alpha_j-\alpha_{j-1}+2} < \ldots < S_{2\alpha_j} = 0
\end{cases}
$$

for $j = 1,2,\ldots,i$; $i = 1,2,\ldots,n$

i.e. a symmetric random walk of 2n steps from (0,0) to (2n,0) has at most one turning point between returns to the x-asis.

In this paper we consider the problem of simple random walk of 2n steps starting from (0,0) and terminating at (2n,0) satisfying condition 'E' and is such that the path does not touch or cross the line $y = k$.

Notation:

For a path$[S_o = 0, S_1, S_2, \ldots, S_{2n} = 0]$ satisfying the condition 'E' we define the following notations:

A-point	:	a point (j, S_j) with $S_j = 0$ i.e. a return to the x-asis.
A^+ (A^-)	:	An A-point s.t., $S_{j-1} = +1(S_{j-1} = -1)$. It is a positive (negative) return point.
V (wave)	:	A segment of a path included between two consecutive A-points. The segment from origin to the first return point is also regarded as a wave.
V^+ (V^-)	:	a wave V s.t. $S_j > 0$ $(S_j < 0)$ at the intervening positions.
V^+ (k^-)	:	a V^+ not reaching the line $y = k$ $(k > 0)$.
B-point	:	a point (j, S_j) of the path with $S_j = 0$ and $S_{j-1} \cdot S_{j+1} = -1$. It is called a crossing or intersection with x-axis.
C (section)	:	a segment of a path included between two consecutive B-points. The segments from origin to the first

		B-point and that from the last B-point to the end point $(2n,0)$ are also regarded as sections.
C^+ (C^-)	:	a section C s.t. $S_j \geqslant 0$ $(S_j \leqslant 0)$ in between.
C_n	:	a path $[S_o, S_1, \ldots, S_n]$ with $S_o = S_n = 0$ and n even.
$C_n(k^-)$	:	a C_n not reaching the line $y = k$.
$C_n(k^-)$	:	a C_n (k^-) with $S_1 = +1$.
$C^+(k^-)$	:	a C^+ not reaching the line $y = k$
$C_n(k_1^-, k_2^-)$	:	a C_n-path which does not reach the line $y = k_1$ and $y = -k_2$ $(k_1, k_2 > 0)$.
$(\ldots)_E$	:	Number of possible paths of the type ... and satisfying condition E.

The following generating functions which are easily determined will be used in the sequel:

$$G_s(V^+(k^-)) \;:\; \sum_{i=1}^{k-1} (pqs^2)^i = \frac{pqs^2 - (pqs^2)^k}{1 - pqs^2} \qquad (1)$$

$$G_s(V^-) \;:\; \sum_{i=1}^{\infty} (pqs^2)^i = \frac{pqs^2}{1 - pqs^2} \qquad (2)$$

Theorem:

$$(C_{2n,r,r_1}^{2b(2h)}(+)(k^-))_E = \binom{r_1 - 1}{b}\binom{r - r_1 - 1}{b - 1}\binom{n - h - 1}{r - r_1 - 1} f(h,k,r_1) \qquad (3)$$

$$(C_{2n,r,r_1}^{2b(2h)}(-)(k^-))_E = \binom{r_1 - 1}{b - 1}\binom{r - r_1 - 1}{b}\binom{n - h - 1}{r - r_1 - 1} f(h,k,r_1) \qquad (4)$$

$$(C_{2n,r,r_1}^{2b-1(2h)}(k^-))_E = 2\binom{r_1 - 1}{b - 1}\binom{r - r_1 - 1}{b - 1}\binom{n - h - 1}{r - r_1 - 1} f(h,k,r_1) \qquad (5)$$

where

$C_{n,r,r_1}^{b(h)}(+)(k^-)$: a C_n (k^-)- path with $S_1 = +1$ and having $b - B$ points, r A-points of which r_1 are A^+ points and h steps above x-axis.

$$f(h,k,r_1) = \sum_{i=0}^{\min(r_1, [\frac{h-r_1}{k-1}])} (-1)^i \binom{r_1}{i} \binom{h-1-(k-1)i}{r_1-1} \tag{6}$$

Proof.

Let $C^{2b(2h)}_{2n,r,r_1}$ (+) (k^-) be a path as envisaged in (3). Since $S_1 > 0$ and $S_{2n-1} > 0$, this path would consist of r_1 V^+ (k^-) constituting (b+1) $C^+(k^-)$ and $(r-r_1)V^-$ forming bC^-. (b+1)C^+ (K^-) can be constructed out of r_1 V^+ (k^-) in $\binom{r_1-1}{b}$ ways and similarly b C^- can be formed out of $(r-r_1)$ V^- in ($b-1$) $\binom{r-r_1-1}{b-1}$ ways. This is akin to distributing δ similar balls into α distinct cells which is possible in $\binom{\beta-1}{\alpha-1}$ ways. Thus, on using (1) and (2) we get the bivariate g.f.

$$G_{s_1,s_2}(C^{2b(2h)}_{2n,r,r_1}(+)(k^-))_E = \Sigma\,\Sigma\,(C^{2b(2h)}_{2n,r,r_1}(+)(k^-))_E\, s_1^{2h}\, s_2^{2n-2h}(pq)^n \tag{7}$$

$$= \binom{r_1-1}{b}\binom{r-r_1-1}{b-1}\left[\frac{pqs_1^2-(pqs_1^2)^k}{1-pqs_1^2}\right]^{r_1} \times \left[\frac{pqs_2^2}{1-pqs_2^2}\right]^{r-r_1}$$

$$= \binom{r_1-1}{b}\binom{r-r_1-1}{b-1}(pqs_1^2)^{r_1}(pqs_2^2)^{r-r_1} \times$$

$$\sum_{i=1}^{r_1}(-1)^i\binom{r_1}{i}(pqs_1^2)^{i(k-1)}\sum_{j=0}^{\infty}\binom{j+r_1-1}{j} \times$$

$$(pqs_1^2)^j\sum_{m=0}^{\infty}\binom{r-r_1+m-1}{m}(pqs_2^2)^m$$

whence the coefficient of $(pq)^n\, s_1^{2h}\, s_2^{2n-2h}$ leads to (3).
Similarly (4) follows.

Proceeding for $(C^{2b-1(2h)}_{2n,r,r_1}(k^-))$, it is obvious that the number of paths remains the same whether $S_1 = +1$ or $S_1 = -1$ which accounts for factor 2 in (5). Such a path has b C^+ (k^-) and b C^- comprising r_1 V^+ (k^-) and $(r-r_1)$ V^-, respectively. Thus arguing as for (3), we get (5).

Letting $k \to \infty$ on (3), (4), (5) we get

$$(C^{2b(2h)}_{2n,r,r_1}(+))_E = \binom{r_1-1}{b}\binom{h-1}{r_1-1}\binom{r-r_1-1}{b-1}\binom{n-h-1}{r-r_1-1} \tag{8}$$

$$(C^{2b(2h)}_{2n,r,r_1}(-))_E = \binom{r_1-1}{b-1}\binom{h-1}{r_1-1}\binom{r-r_1-1}{b}\binom{n-h-1}{r-r_1-1} \tag{9}$$

$$(C^{2b-1(2h)}_{2n,r,r_1})_E = 2\binom{r_1-1}{b-1}\binom{h-1}{r_1-1}\binom{r-r_1-1}{b-1}\binom{n-h-1}{r-r_1-1} \tag{10}$$

where

$C^{b(h)}_{n,r,r_1}$: a C_n-path with b B-points, r A points of which r_1 are A^+-points and h steps above the x-axis.

(8), (9), (10) verify authors' results (1), (2), (3) See [2].

Deductions:

(i) Let $C^{b(h)}_{2n,.,r_1}(+)(k^-)$: A $C_n(k^-)$-path with $S_1 = +1$ and having b B-points, r_1 A^+-*points* and h steps above x-axis.

$$G_{s_1,s_2}(C^{2b(2h)}_{2n,.,r_1}(+)(k^-))_E = \binom{r_1-1}{b}\left[\frac{pqs_1^2-(pqs_1^2)^k}{1-pqs_1^2}\right]^{r_1}\left[\frac{pqs_2^2}{1-pqs_2^2}\right]^{b} \tag{11}$$

$$= \binom{r_1-1}{b}(pqs_1^2)^{r_1}(pqs_2^2)^b \sum_{i=0}^{r_1}\binom{r_i}{i}(-pqs_1^2)^i$$

$$\times \sum_{j=0}^{\infty} \binom{j+r_1-1}{j} (pqs_1^2)^j \sum_{m=0}^{\infty} \binom{b+m-1}{m} (pqs_2^2)^m$$

whence the coefficient of $(pq)^n \; s_1^{2h} \; s_2^{2n-2h}$ gives

$$(C_{2n,.,r_1}^{2b(2h)} (+) (k^-))_E = 2^{n-h-b} \binom{r_1-1}{b} \binom{n-h-1}{b-1} f(h,k,r_1) \quad (12)$$

similarly summing (4) over $b + r_1 + 1 \leqslant r \leqslant \infty$ and (5) over $b+r_1 \leqslant r \leqslant \infty$ we get respectively

$$(C_{2n,.,r_1}^{2b(2h)} (-) (k^-))_E = 2^{n-h-b-1} \binom{r_1-1}{b-1} \binom{n-h-1}{b} f(h,k,r_1) \quad (13)$$

$$(C_{2h,.,r_1}^{2b-1(2h)} (+) (k^-))_E = 2^{n-h-b} \binom{r_1-1}{b-1} \binom{n-h-1}{b-1} f(h,k,r_1) \quad (14)$$

$$= (C_{2n,.,r_1}^{2b-1(2h)} (-) (k^-))_E$$

(ii) Putting $s_1 = s_2 = s$ in (7) we get,

$$G_s(C_{2n,r,r_1}^{2b} (+) (k^-))_E = \binom{r_1-1}{b} \binom{r-r_1-1}{b-1} [1-(pqs^2)^{k-1}]^{r_1}$$

$$\times \left[\frac{pqs^2}{1-pqs^2} \right]^r \quad (15)$$

$$= \binom{r_1-1}{b} \binom{r-r_1-1}{b} \sum_{i=0}^{r_1} \binom{r_1}{i} (pqs^2)^{(k-1)i}$$

$$\times \sum_{j=0}^{\infty} \binom{r_1+j-1}{j} (pqs^2)^{r+j}$$

where

$C_{n,r,r_1}^{b} (+) (k^-)$: a $C_n (+) \; (k^-)$-path with b B-points, r A-points of which r_1 are A^+-points.

The coefficient of $(pqs^2)^n$ in (15) gives

$$(C^{2b}_{2n,r,r_1}\ (+)\ (k^-))_E = \binom{r_1-1}{b}\binom{r-r_1-1}{b-1} \sum_{i=0}^{\min(r_1,|\frac{n-r}{k-1}|)} (-1)\binom{r_1}{i}$$

$$\text{x}\binom{n-1-(k-1)i}{r-1} \qquad (16)$$

$$= \binom{r_1-1}{b}\binom{r-r_1-1}{b-1} f(n,k,r,r_1)$$

where

$$f(n,k,r,r_1) = \sum_{i=0}^{\min(r_1,|\frac{n-r}{k-1}|)} (-1)^i \binom{r_1}{i}\binom{n-1-(k-1)i}{r-1}$$

Similarly summing (4) and (5) over h we get

$$(C^{2b}_{2n,r,r_1}\ (-)\ (k^-))_E = \binom{r_1-1}{b-1}\binom{r-r_1-1}{b} f(n,k,r,r_1) \qquad (17)$$

$$(C^{2b-1}_{2n,r,r_1}\ (k^-))_E = 2\binom{r_1-1}{b-1}\binom{r-r_1-1}{b-1} f(n,k,r,r_1) \qquad (18)$$

For $k \to \infty$ in (16), (17), (18) we get

$$(C^{2b}_{2n,r,r_1}\ (+))_E = \binom{r_1-1}{b}\binom{r-r_1-1}{b-1}\binom{n-1}{r-1} \qquad (19)$$

$$(C^{2b}_{2n,r,r_1}\ (-))_E = \binom{r_1-1}{b-1}\binom{r-r_1-1}{b}\binom{n-1}{r-1} \qquad (20)$$

$$(C^{2b-1}_{2n,r,r_1})_E = 2\binom{r_1-1}{b-1}\binom{r-r_1-1}{b-1}\binom{b-1}{r-1} \qquad (21)$$

where

C^b_{n,r,r_1} : a C_n-path with b B-points, r A-points of which r_1 are A^+-points.

(iii) $C^h_{n,r,r_1}(k^-)\,(C^h_{n,r,r_1})$: a $C_n(k^-)$ (C_n)-path with r A-points, r_1 $A+$ points and h steps above the x-axis.

Summing (3) over $1 \leqslant b \leqslant \min(r_1-1, r-r_1)$; (4) over $1 \leqslant b \leqslant \min(r_1, r-r_1-1)$ and (5) over $1 \leqslant b \leqslant \min(r_1, r-r_1)$ respectively we get

$$(C^{2h}_{2n,r,r_1}(+)(k^-))^e_E = \binom{r-2}{r_1-2}\binom{n-h-1}{r-r_1-1} f(h,k,r_1) \quad (22)$$

$$(C^{2h}_{2n,r,r_1}(-)(k^-))^e_E = \binom{r-2}{r_1}\binom{n-h-1}{r-r_1-1} f(h,k,r_1) \quad (23)$$

$$(C^{2h}_{2n,r,r_1}(k))^0_E = 2\binom{r-2}{r_1-1}\binom{n-h-1}{r-r_1-1} f(h,k,r_1) \quad (24)$$

where superscript 'e' ('o') denotes the even (odd) number of crossings.

Adding (22), (23) and (24) we get

$$(C^{2h}_{2n,r,r_1}(k))_E = \binom{r}{r_1}\binom{n-h-1}{r-r_1-1} f(h,k,r_1) \quad (25)$$

Letting $k \to \infty$ in (25) we get

$$(C^{2h}_{2n,r,r_1})_E = \binom{r}{r_1}\binom{n-h-1}{r-r_1-1}\binom{h-1}{r_1-1} \quad (26)$$

(iv) Summing (11) over $b+1 \leqslant r_1 \leqslant h$; (13) and (14) over $b \leqslant r_1 \leqslant h$ we get

$$(C^{2b(2h)}_{2n}(+)(k))_E = 2^{n-h-b}\binom{n-h-1}{b-1}\sum_{r_1=b+1}^{h}\binom{r_1-1}{b} f(h,k,r_1) \quad (27)$$

$$(C^{2b(2h)}_{2n}(-)(k))_E = 2^{n-h-b-1}\binom{n-h-1}{b}\sum_{r_1=b}^{h}\binom{r_1-1}{b-1} f(h,k,r_1) \quad (28)$$

$$(C^{2b-1(2h)}_{2n}(k))_E = 2^{n-h-b+1}\binom{n-h-1}{b-1}\sum_{r_1=b}^{h}\binom{r_1-1}{b-1} f(h,k,r_1) \quad (29)$$

where

$C_n^{b(h)}(k)$: a $C_n(k)$-path with b B-points and h steps above the x-axis.

(v) Summing (16) and (19) over $b+1 \leqslant r_1 \leqslant r-b$; (17) and (20) over $b \leqslant r_1 \leqslant r-b-1$; (18) and (21) over $b \leqslant r_1 \leqslant r-b$ we get respectively

$$(C_{2n,r}^{2b}\ (+)\ (\mathrm{k})_E = \sum_{r_1=b+1}^{r-b} \binom{r_1-1}{b}\binom{r-r_1-1}{b-1} f(n,k,r,r_1) \quad (30)$$

$$(C_{2n,r}^{2b}(+))_E = \binom{n-1}{r-1} \binom{r-1}{2b} \quad (31)$$

$$(C_{2n,r}^{2b}(-)\ (k))_E = \sum_{r_1=b}^{r-b-1} \binom{r_1-1}{b-1}\binom{r-r_1-1}{b} f(n,k,r,r_1) \quad (32)$$

$$(C_{2n,r}^{2b}(-))_E = \binom{n-1}{r-1} \binom{r-1}{2b} \quad (33)$$

$$(C_{2n,r}^{2b-1}(k))_E = 2 \sum_{r_1=b}^{r-b} \binom{r_1-1}{b-1}\binom{r-r_1-1}{b} f(n,k,r,r_1) \quad (34)$$

$$(C_{2n,r}^{2b-1}\ (+))_E = \binom{n-1}{r-1}\binom{r-1}{2b-1} = C_{2n,r}^{2b-1}(-))_E \quad (35)$$

where

$C_{n,r}^{b}\ (k)\ (C_{n,r}^{b})$: a $C_n(k)$ (C_n)-path with b B-points and r A-points.

Clearly from (31), (33) and (35) we get

$$(C_{2n,r}^{b}\ (+))_E = \binom{n-1}{r-1}\binom{r-1}{r-1} = C_{2n,r}^{b}\ (-))_E \quad (36)$$

verifying authors' result (15) see [1].

$$(C^b_{2n}(+))_E = 2^{h-1-b}\binom{n-1}{b} = (C^b_{2n}(-))_E \tag{37}$$

where

C^b_n : a C_n-path with b B-points

Summing (25) over r, r_1 and h we get

$$(C_{2n}(k))_E = \sum_{r_1}^{n} \sum_{r=r_1} \sum_{h=r_1}^{n-r+r_1} \binom{r}{r_1}\binom{n-h-1}{r-r_1-1} f(h,k,r_1) \tag{38}$$

Letting $k \to \infty$ in (38) we get

$$(C_{2n})_E = 2.3^{n-1} \tag{39}$$

verifying authors' result (20) See [2].

In the case of two absorbing barriers we can easily show that:

Theorem:

Let $C^{b(h)}_{n,r,r_1}(k_1,k_2)$: a $C_n(k_1,k_2)$-path with b-B points, r A-points of which r_1 A^+-points.

$$(C^{2b(2h)}_{2n,r,r_1}(+)(k_1,k_2))_E = \binom{r_1-1}{b}\binom{r-r_1-1}{b-1} f_1(h,k_1,r_1) \times f_2(n,h,k_2,r,r_1) \tag{40}$$

$$(C^{2b(2h)}_{2n,r,r_1}(-)(k_1,k_2))_E = \binom{r_1-1}{b-1}\binom{r-r_1-1}{b} f(h,k_1,r_1) \times f_2(n,h,k_2,r,r_1) \tag{41}$$

$$(C^{2b-1(2h)}_{2n,r,r_1}(k_1,k_2))_E = 2\binom{r_1-1}{b-1}\binom{r-r_1-1}{b-1} f_1(h,k_1,r_1) \times f_2(n,h,k_2,r,r_1) \tag{41}$$

where

$$f_1(h,k_1,r_1) = \sum_{i=0}^{\min(r_1,[\frac{h-r_1}{k_1-1}])} (-1)^i \binom{r_1}{i}\binom{h-1-(k_1-1)i}{r_1-1} \quad (43)$$

$$f_2(n,h,k_2,r,r_1) = \sum_{j=0}^{\min(r_1,[\frac{n-h-r+r_1}{k_2-1}])} (-1)^j \binom{r-r_1}{j}\binom{n-h-1-(k_2-1)j}{r-r_1-1} \quad (44)$$

It is easily seen that for $k_2 \to \infty$; (40) to (42) lead to (3) to (5), respectively, of Theorem (1).

References

[1] Feller, W: An Introduction to Probability Theory and its application, Vol. 1 (3rd Ed.) John Wiley, N.Y.

[2] Sushma Sindwani and Kanwar Sen: "On Fluctuations In Coin Tossing and Random Walk", J. Korean Math. Soc. Vol. 13, No. 1, pp. 105-111.